THE COLLINS-ROSCOE MECHANISM AND D-SPACES

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Abstract. We prove that if a space X is well ordered (αA), or linearly semi-stratifiable, or elastic then X is a D-space.

1. Introduction

The connections between D-spaces and generalized metric spaces has been extensively studied. The aim of this paper is to prove the following theorems.

- Spaces satisfying well-ordered (αA) are D-spaces.
- Linearly semi-stratifiable spaces are D-spaces.
- Elastic spaces are D-spaces.

The proofs are based on Gruenhage's method of *sticky relations*.

The paper has the following structure. In Section 2 we introduce the Collins-Roscoe mechanism and give the basic definitions. In Section 3 we define the notion of D-spaces and briefly introduce how sticky relations are used to prove that a certain space is D. In Sections 4, 5 and 6 we prove the three results above.

2. The Collins-Roscoe mechanism

The expression *Collins-Roscoe structuring mechanism* refers to several definitions of generalized metric properties. In [5] P. J. Collins and A. W. Roscoe introduced the following notion.

Definition 2.1. *We say that* a space X satisfies condition (G) *iff there is* $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$, *where* $\mathcal{W}(x) = \{W(m,x) : m \in \omega\}$, *such that* $x \in W(m,x) \subseteq X$ *with the following property. For every open set* U *containing* $x \in X$, *there exists an open set* $V(x,U)$ *containing* x *such that* $y \in V(x,U)$ *implies* $x \in W(m,y) \subseteq U$ *for some* $m \in \omega$.

If we strengthen condition (G) by not allowing the natural number m to vary with y, then we say that X *satisfies condition* (A). The precise definition is the following.

Definition 2.2. *We say that* a space X satisfies condition (A) *iff there is* $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$, *where* $\mathcal{W}(x) = \{W(m,x) : m \in \omega\}$, *such that* $x \in W(m,x) \subseteq X$ *with the following property. For every open set* U *containing* $x \in X$, *there exists*

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an open set $V(x,U)$ containing x and a natural number $m = m(x,U)$ such that $x \in W(m,y) \subseteq U$ for all $y \in V(x,U)$.

If each $W(n,x)$ is open (a neighborhood of x), we say that X *satisfies open (neighborhood)* (G) *or open(neighborhood)* (A), respectively. If $W(n+1,x) \subseteq W(n,x)$ for each $n \in \omega$, we say that X *satisfies decreasing* (G) *or decreasing* (A).

The Collins-Roscoe mechanism has been extensively studied, and a lot of significant results have been obtained. Let us summarize [5, Theorem 1] and [6, Theorem 8] in 2.3.

Theorem 2.3. *The following are equivalent for a space X.*
(1) X is metrisable,
(2) X satisfies decreasing open (A),
(3) X satisfies decreasing open (G),
(4) X satisfies decreasing neighborhood (A).

Stratifiable spaces are well known generalizations of metric spaces; see [14]. They have a characterization using the Collins-Roscoe mechanism as well. Theorem 2.4 summarizes [1, Theorem 2.2] and a remark from [6].

Theorem 2.4. *The following are equivalent for a space X.*
(1) X is stratifiable,
(2) X satisfies decreasing (G) and has countable pseudo-character,
(3) X satisfies decreasing (A) and has countable pseudo-character.

We define a third condition denoted by *(F)*, which is weaker than condition (G).

Definition 2.5. *We say that* a space X satisfies condition (F) *iff there is $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$ such that $x \in W \subseteq X$ for all $W \in \mathcal{W}(x)$ with the following property. For every open U containing x there is an open $V = V(x,U)$ containing x such that $y \in V$ implies $x \in W \subseteq U$ for some $W \in \mathcal{W}(y)$.*

We say that X *satisfies well-ordered* (F) if each $\mathcal{W}(x)$ is well-ordered by reverse inclusion. We make a remark about well ordered (F) spaces in the last section.

3. D-spaces and sticky relations

In [7], van Douwen and Pfeffer introduced the concept of D-spaces.

Definition 3.1. *An* open neighborhood assignment (ONA) *on a space (X,τ) is a function $N : X \to \tau$ such that $x \in N(x)$ for all $x \in X$. X is a D-space iff for all ONA N, there is a closed discrete subset D of X such that $N[D] = \{N(d) : d \in D\}$ covers X.*

We recommend G. Gruenhage's paper [13] which gives a full review on what we know and do not know about D-spaces.

The following method of G. Gruenhage [12] provides us a useful tool for proving that a spaces is a D-space.

Definition 3.2. *Let X be a space. A relation R on X is* nearly good *iff $x \in \overline{A}$ implies that there is $y \in A$ such that xRy. Let N denote an ONA. If $X' \subseteq X$ and $D \subseteq X$ we say that D* is N-sticky mod R on X' *if whenever $x \in X'$ and xRy for some $y \in D$ then $x \in \cup N[D]$.*

Theorem 3.3 ([12, Proposition 2.2])**.** *Let X be a space and N an ONA on X. Suppose R is a nearly good relation on X such that every non-empty closed subset F of X contains a non-empty closed discrete subset D which is N-sticky mod R on F. Then there is a closed discrete D^* in X with $\cup N[D^*] = X$.*

Let $Z \subseteq X$ and N an ONA on X. We say that Z *is N-close* iff $Z \subseteq N(x)$ for all $x \in Z$.

Theorem 3.4 ([12, Proposition 2.4])**.** *Let N be a neighborhood assignment on X. Suppose there is a nearly good R on X such that for any $y \in X$, $R^{-1}(y) \setminus N(y)$ is the countable union of N-close sets. Then there is a closed discrete D such that $\cup N[D] = X$.*

As an easy application of his method, Gruenhage proves in [12, Proposition 2.5] that spaces satisfying open (G) are D-spaces. The same proof yields the following.

Proposition 3.5. *If the space X satisfies condition (G) then X is a D-space.*

Proof. Let $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$, where $\mathcal{W}(x) = \{W(n, x) : n \in \omega\}$, witness condition (G). We use the notation $V(x, U)$ from Definition 2.1 as well.

Let N be an ONA on X. We will apply Theorem 3.4 for the following relation R. Let xRy iff $x \in W(n, y) \subseteq N(x)$ for some $n \in \omega$. Then R is nearly good; indeed, let $x \in \overline{A}$ for some $A \subseteq X$. Then $V(x, N(x)) \cap A \neq \emptyset$ and xRy for any $y \in V(x, N(x)) \cap A$.

Let $y \in X$ and let $C_n = \{x \in X : x \in W(n, y) \subseteq N(x)\}$ for $n \in \omega$. Clearly $R^{-1}(y) = \cup\{C_n : n \in \omega\}$ and $C_n \subseteq W(n, y)$ is N-close. Thus by Theorem 3.4 there is some closed discrete $D \subseteq X$ such that $X = \cup N[D]$. $\square$

4. Well ordered (αA) spaces

Our goal now is to prove that spaces satisfying well-ordered (αA) are D-spaces.

Definition 4.1. *Let X be a space, α an ordinal. We say that X* satisfies (αA) *iff there is $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$, where $\mathcal{W}(x) = \{W(\beta, x) : \beta < \alpha\}$, such that $x \in W(\beta, x) \subseteq X$ with the following property. For every open U containing x, there exists an open set $V(x, U)$ containing x and an ordinal $\beta = \varphi(x, U) < \alpha$ such that*

$$x \in W(\beta, y) \subseteq U \text{ for all } y \in V(x, U).$$

If, in addition, $W(\beta, x) \subseteq W(\gamma, x)$ whenever $\gamma < \beta < \alpha$, then we say that X satisfies well-ordered (αA).

Theorem 4.2. *If the space X satisfies well-ordered (αA) (for some ordinal α) then X is a D-space.*

Proof. Let $\mathcal{W} = \{\mathcal{W}(x) : x \in X\}$, where $\mathcal{W}(x) = \{W(\beta, x) : \beta < \alpha\}$, witness condition (αA). We use the notation $V(x, U)$ and $\varphi(x, U)$ from Definition 4.1 as well.

Let N be a neighborhood assignment on X. We will define a relation R on X and apply Theorem 3.3. Let xRy iff $x \in W(\beta, y)$ for $\beta = \varphi(x, N(x))$. Clearly R is nearly good; indeed, let $x \in \overline{A}$ for some $A \subseteq X$. Then $V(x, N(x)) \cap A \neq \emptyset$ and xRy for any $y \in V(x, N(x)) \cap A$.

Suppose that $F \subseteq X$ is closed and non-empty. We show that there is a closed discrete $D \subseteq F$ such that D is N-sticky mod R on F. Let $\beta_0 = \min\{\varphi(y, N(y)) :$

$y \in F\}$ and pick $y \in F$ such that $\beta_0 = \varphi(y, N(y))$. Let $D = \{y\}$. Suppose that xRy for some $x \in F$. Then for $\beta = \varphi(x, N(x))$ the following holds

$$x \in W(\beta, y) \subseteq W(\beta_0, y) \subseteq N(y)$$

since $\beta \geq \beta_0$. Thus D is N-sticky mod R on F, and so by Theorem 3.3 there is some closed discrete $D^* \subseteq X$ such that $X = \cup N[D^*]$. □

Now we formulate some corollaries. It is proved in [2] that (semi-)stratifiable spaces are D-spaces. We can slightly strengthen this result.

Definition 4.3 ([20, Definition 2.2])**.** *Let (X, τ) be a T_1 topological space and $\alpha \geq \omega$ an ordinal. X is said to be* stratifiable over α *or* linearly stratifiable *iff there exists a mapping $G : \alpha \times \tau \to \tau$ with the following properties (write $U_\beta = G(\beta, U)$).*

(1) $\overline{U_\beta} \subseteq U$ for all $\beta < \alpha$ and $U \in \tau$,
(2) $\bigcup\{U_\beta : \beta < \alpha\} = U$ for all $U \in \tau$,
(3) if $U \subseteq V$ then $U_\beta \subseteq V_\beta$ for all $\beta < \alpha$,
(4) if $\gamma < \beta < \alpha$ then $U_\gamma \subseteq U_\gamma$ for all $U \in \tau$.

From [17, Theorem 5.2] we know that linearly stratifiable spaces are well-ordered (αA), thus we have the following.

Corollary 4.4. *Linearly stratifiable spaces are D-spaces.*

For a space (X, τ) let $\mathcal{D}_X = \{(x, U) : x \in U \in \tau\}$.

Definition 4.5. *A space (X, τ) is said to be* Borges normal *iff there are operators $H : \mathcal{D}_X \to \tau$ and $n : \mathcal{D}_X \to \omega$ such that $H(x, U) \cap H(y, V) \neq \emptyset$ and $n(x, U) \leq n(y, V)$ implies $y \in U$ for all $(x, U), (y, V) \in \mathcal{D}_X$.*

It can be proved that Borges normal spaces are special well-ordered (αA) spaces.

Theorem 4.6 ([18, Theorem 2.1])**.** *A space X is Borges normal iff X satisfies well-ordered (ωA).*

Corollary 4.7. *Borges normal spaces are D-spaces.*

5. Linearly semi-stratifiable spaces

In [2], Borges and Wehrly proved that semi-stratifiable spaces are D-spaces. We find a common generalization of this and Corollary 4.4, that is, we show that linearly semi-stratifiable spaces are D-spaces.

Let (X, τ) be a T_1-space and let $\mathcal{F}_X$ denote the family of all closed subsets of X.

Definition 5.1. *X is said to be* semi-stratifiable over α *(for some ordinal α) or* linearly semi-stratifiable *if there exists a mapping $F : \alpha \times \tau \to \mathcal{F}_X$ such that:*
(1) $U = \cup\{F(U, \beta) : \beta < \alpha\}$ for all $U \in \tau$;
(2) if $U \subseteq W$ then $F(U, \beta) \subseteq F(W, \beta)$ for all $\beta < \alpha$;
(3) if $\gamma < \beta < \alpha$, then $F(U, \gamma) \subseteq F(U, \beta)$ for all $U \in \tau$.

Theorem 5.2. *If the space X is semi-stratifiable over α (for some ordinal α) then X is a D-space.*

Proof. Let $F : \alpha \times \tau \to \mathcal{F}_X$ be the function witnessing that X is linearly semi-stratifiable.

Let N be ONA on X. We will define a relation R on X and apply Theorem 3.3. Let $\sigma(x) = \min\{\beta < \alpha : x \in F(N(x), \beta)\}$ for $x \in X$. Let xRy iff $x \in N(y)$

or $\sigma(x) < \sigma(y)$. We prove that R is nearly good. Suppose that $x \in \overline{A}$ however $x \notin R^{-1}(y)$ for all $y \in A$. Thus $x \notin \cup\{N(y) : y \in A\}$ and $\sigma(y) \leq \sigma(x)$ for all $y \in A$. Thus $y \in F(N(y), \sigma(y)) \subseteq F(N(y), \sigma(x))$ for all $y \in A$. Thus

$$A \subseteq F(\cup\{N(y) : y \in A\}, \sigma(x)) \subseteq \cup\{N(y) : y \in A\} \subseteq X \setminus \{x\}.$$

$F(\cup\{N(y) : y \in A\}, \sigma(x))$ is closed hence $x \in \overline{A} \subseteq F(\cup\{N(y) : y \in A\}, \sigma(x))$, which is a contradiction. This proves that R is nearly good.

Suppose that $F \subseteq X$ is closed and nonempty. We show that there is a closed discrete $D \subseteq F$ such that D is N-sticky mod R on F. Let $\sigma = \min\{\sigma(y) : y \in F\}$ and let $y \in F$ such that $\sigma = \sigma(y)$. Let $D = \{y\}$. If xRy for some $x \in F$ then $x \in N(y)$ since $\sigma(x) \geq \sigma(y)$. Thus D is N-sticky mod R on F, and so by Theorem 3.3 there is some closed discrete $D^* \subseteq X$ such that $X = \cup N[D^*]$. □

6. Elastic spaces

Our aim now is to prove that *elastic spaces* are D-spaces. Elastic spaces were first introduced by H. Tamano and J. E. Vaughan in [19] as a natural generalization of stratifiable spaces. First we need the definition of a *pair-base* which is due to J. G. Ceder [4].

Definition 6.1. *Let X be a space. A collection $\mathcal{P}$ of ordered pairs $P = (P_1, P_2)$ of subsets of X is called a* pair-base *provided that P_1 is open for all $P \in \mathcal{P}$ and that for every $x \in X$ and open set U containing x, there is a $P \in \mathcal{P}$ such that $x \in P_1 \subseteq P_2 \subseteq U$.*

The following definition of elastic spaces is an improvement of the original one and due to Gartside and Moody [11].

Definition 6.2. *A space X is* elastic *if there is a pair-base $\mathcal{P}$ on X and transitive relation $\leq$ on $\mathcal{P}$ such that*
(1) if $P, P' \in \mathcal{P}$ are such that $P_1 \cap P'_1 \neq \emptyset$ then $P \leq P'$ or $P' \leq P$;
(2) if $P \in \mathcal{P}$ and $\mathcal{P}' \subseteq \{P' \in \mathcal{P} : P' \leq P\}$ then $\overline{\cup\{P'_1 : P' \in \mathcal{P}'\}} \subseteq \cup\{P'_2 : P' \in \mathcal{P}'\}$.

Note that the relation $\leq$ should be reflexive.

Before we show that elastic spaces are D-spaces, we need the following proposition which is implicitly in [19, Lemma 2].

Proposition 6.3. *Suppose that $\leq$ is a reflexive, transitive relation on the set S, then there is a reflexive, antisymmetric relation $\preceq$ on S such that:*
(1) if $x, y \in S$ and $x \leq y$, then $x \preceq y$ or $y \preceq x$;
(2) if A is a non-empty subset of S, then A has a $\preceq$-minimal element, i.e. there is an $x \in A$ such that $y \not\preceq x$ whenever $y \in A \setminus \{x\}$;
(3) if $A \subseteq S$ and $A \subseteq \{x \in S : x \preceq s\}$ for some $s \in S$, then $A \subseteq \{x \in S : x \leq s'\}$ for some $s' \in S$.

Proof. Let $S = \{s_\alpha : \alpha < \kappa\}$ and $S(\alpha) = \{s \in S : s \leq s_\alpha\}$ for $\alpha < \kappa$. By induction on $\alpha < \kappa$ we define a reflexive and antisymmetric relation $\preceq_\alpha$ on $\cup\{S(\beta) : \beta \leq \alpha\}$ such that $\preceq_\alpha$ extends $\preceq_\beta$ for $\beta < \alpha < \kappa$ and then let $\preceq$ to be $\cup\{\preceq_\alpha : \alpha < \kappa\}$.

Let $\preceq_0$ denote a well-ordering of $S(0)$. Let $\alpha < \kappa$ and suppose that $\preceq_\beta$ is constructed for $\beta < \alpha$. Let $S'(\alpha) = S(\alpha) \setminus \cup\{S(\beta : \beta < \alpha)\}$. Let $\leq_\alpha$ be a well-ordering on $S'(\alpha)$ and also put $s \leq_\alpha s'$ if $s' \in S'(\alpha)$ and $s \in S(\alpha) \cap (\cup\{S(\beta) : \beta < \alpha\})$. Let $\preceq_\alpha = \cup\{\preceq_\beta : \beta < \alpha\} \cup \leq_\alpha$; this is reflexive and antisymmetric. Finally, let $\preceq$ to be $\cup\{\preceq_\alpha : \alpha < \kappa\}$.

Clearly $\preceq$ is reflexive and antisymmetric on S. First we shall verify (1). Let us suppose that $x \le y$ for some $x, y \in S$. Let $\alpha_0 = \min\{\alpha < \kappa : y \in S(\alpha)\}$, since $\le$ is transitive we have $x \in S(\alpha_0)$. Then by definition $x \le_\alpha y$ or $y \le_\alpha x$ thus $x \preceq y$ or $y \preceq x$.

Next we show that every nonempty $A \subseteq S$ has a $\preceq$-minimal element. First note the following.

Claim 6.4. *If $s, s' \in S$, $s \preceq s'$ and $\alpha < \kappa$ is minimal such that $s, s' \in \cup\{S(\beta) : \beta \le \alpha\}$ then $s \preceq_\alpha s'$, $s' \in S'(\alpha)$ and $s \in S(\alpha)$.*

Proof. Let $\gamma < \kappa$ minimal such that $s \preceq_\gamma s'$. Thus $s, s' \in \cup\{S(\beta) : \beta \le \gamma\}$ hence $\alpha \le \gamma$. If $\alpha < \gamma$ then $s, s' \notin S'(\gamma)$ so s and s' are not related by $\le_\gamma$. Hence there is some $\beta < \gamma$ such that $s \preceq_\beta s'$ (by the definition of $\preceq_\gamma$). This contradicts the choice of γ. Thus $\alpha = \gamma$ and $s \preceq_\alpha s'$. Clearly $s \le_\alpha s'$ by the definition of $\preceq_\alpha$ since s and s' are not related by $\preceq_\delta$ for any $\delta < \alpha = \gamma$. Thus $s' \in S'(\alpha)$ and $s \in S(\alpha)$. $\square$

Suppose that $\emptyset \ne A \subseteq S$. Let $\alpha_0 = \min\{\alpha < \kappa : A \cap S(\alpha) \ne \emptyset\}$. Then $A \cap S(\alpha_0) \subseteq S'(\alpha_0)$. Since $S'(\alpha_0)$ is well-ordered by $\le_{\alpha_0}$, there is an $x \in A \cap S(\alpha_0)$ which is $\le_{\alpha_0}$-minimal in $A \cap S(\alpha_0)$. We show that x is $\preceq$-minimal in A. Clearly x is $\preceq$-minimal in $A \cap S(\alpha_0)$. If $y \preceq x$ for some $y \in A$ then for the minimal $\alpha < \kappa$ such that $x, y \in \cup\{S(\beta) : \beta \le \alpha\}$ we have $\alpha_0 < \alpha$. By the claim $x \in S'(\alpha)$ which is a contradiction. Thus x is $\preceq$-minimal in A, i.e. (2) holds.

Finally we show that if A is $\preceq$ upper bounded then also $\le$ upper bounded. Suppose that $A \subseteq \{x \in S : x \preceq s\}$ for some $s \in S$. Let $\alpha_0 = \min\{\alpha < \kappa : s \in S(\alpha)\}$. We shall show that $A \subseteq S(\alpha_0)$, that is, s_{α_0} is a $\le$ upper bound for A. Clearly $s \in S'(\alpha_0)$. Let $x \in A$ and let α be minimal such that $x, s \in \cup\{S(\beta) : \beta \le \alpha\}$. Then $s \in S'(\alpha)$ by the claim and $x \preceq s$. Hence $\alpha = \alpha_0$ and $x \in S(\alpha_0)$, using the claim again. This proves $A \subseteq S(\alpha_0)$. $\square$

Theorem 6.5. *If X is elastic then X is a D-space.*

Proof. Let $\mathcal{P}$ be the pair-base on X with some relation $\le$ witnessing that X is elastic. There is a reflexive antisymmetric relation $\preceq$ on $\mathcal{P}$ by Proposition 6.3 with the following properties:

(a) if $P, P' \in \mathcal{P}$ are such that $P_1 \cap P'_1 \ne \emptyset$ then $P \preceq P'$ or $P' \preceq P$;
(b) if $\mathcal{P}'$ is a non-empty subset of $\mathcal{P}$, then there is a $\preceq$-minimal element of $\mathcal{P}'$;
(c) if $P \in \mathcal{P}$ and $\mathcal{P}' \subseteq \{P' \in \mathcal{P} : P' \preceq P\}$ then $\overline{\cup\{P'_1 : P' \in \mathcal{P}'\}} \subseteq \cup\{P'_2 : P' \in \mathcal{P}'\}$.

Let us enumerate $\mathcal{P}$ as follows. By property (b), there is an element of $\mathcal{P}$, denoted by P^0, such that $P \not\preceq P^0$ whenever $P \in \mathcal{P} \setminus \{P^0\}$. Assume P^γ has been selected for each $\gamma < \beta$, and $P \not\preceq P^\gamma$ whenever $P \in \mathcal{P} \setminus \{P^\eta : \eta \le \gamma\}$. If $\mathcal{P} \setminus \{P^\gamma : \gamma < \beta\} \ne \emptyset$, there is an element of $\mathcal{P} \setminus \{P^\gamma : \gamma < \beta\}$, denoted by P^β, such that $P \not\preceq P^\beta$ whenever $P \in \mathcal{P} \setminus \{P^\gamma : \gamma \le \beta\}$. Thus $\mathcal{P}$ can be enumerated as $\mathcal{P} = \{P^\beta : \beta < \lambda\}$ such that

(d) $P^{\beta'} \not\preceq P^\beta$ if $\beta < \beta' < \lambda$.

Let N be an ONA on X. We will define a relation R on X and apply Theorem 3.3. Let $\sigma(x) = \min\{\beta < \lambda : x \in P^\beta_1 \subseteq P^\beta_2 \subseteq N(x)\}$ for $x \in X$. Let xRy iff $x \in N(y)$ or $P^{\sigma(x)} \preceq P^{\sigma(y)}$. We prove that R is nearly good. Suppose that $x \in \overline{A}$ however $x \notin R^{-1}(y)$ for all $y \in A$. Thus $x \notin N(y)$ and $P^{\sigma(x)} \not\preceq P^{\sigma(y)}$ for all $y \in A$. Let $A_1 = A \cap P^{\sigma(x)}_1 \ne \emptyset$. Since $P^{\sigma(x)}_1 \cap P^{\sigma(y)}_1 \ne \emptyset$ we have $P^{\sigma(y)} \preceq P^{\sigma(x)}$ for all $y \in A_1$. Thus

$$\overline{\cup\{P^{\sigma(y)}_1 : y \in A_1\}} \subseteq \cup\{P^{\sigma(y)}_2 : y \in A_1\} \subseteq \cup\{N(y) : y \in A_1\} \subseteq X \setminus \{x\}$$

using $P_2^{\sigma(y)} \subseteq N(y)$ and $x \notin N(y)$ for $y \in A_1$. Clearly $x \in \overline{A_1}$ and

$$A_1 \subseteq \cup\{P_1^{\sigma(y)} : y \in A_1\}.$$

This yields $x \in \overline{A_1} \subseteq X \setminus \{x\}$ which is a contradiction. Thus R is nearly good.

Suppose that $F \subseteq X$ is closed and nonempty. We show that there is a closed discrete $D \subseteq F$ such that D is N-sticky mod R on F. Let $\sigma = \min\{\sigma(y) : y \in F\}$ and let $y \in F$ such that $\sigma = \sigma(y)$. Let $D = \{y\}$. Suppose xRy for some $x \in F$. If $P^{\sigma(x)} \preceq P^{\sigma(y)}$ then $\sigma(x) = \sigma(y)$ since $\sigma(y) \leq \sigma(x)$ (and by property (d)). Thus $x \in P_1^{\sigma(x)} = P_1^{\sigma(y)} \subseteq P_2^{\sigma(y)} \subseteq N(y)$. If $P^{\sigma(x)} \not\preceq P^{\sigma(y)}$ then $x \in N(y)$. Thus D is N-sticky mod R on F, and so by Theorem 3.3 there is some closed discrete $D^* \subseteq X$ such that $X = \cup N[D^*]$. □

Proto-metrisable spaces were introduced by P. Nyikos in his study of nonarchimedian spaces in [16].

Definition 6.6. *Let X be a space, $\mathcal{B}$ a base for the topology. The base $\mathcal{B}$ is said to be an* orthobase *if whenever $\mathcal{B}' \subseteq \mathcal{B}$, either $\cap\mathcal{B}'$ is open or $\mathcal{B}'$ is a local base for any point in $\cap\mathcal{B}'$. A space is said to be* proto-metrisable *if it is paracompact and has an orthobase.*

Gartside and Moody proved that proto-metrisable spaces are elastic [10, Corollary 9]. Thus we can deduce the following corollary, which had already been obtained by Borges and Wehrly in [3].

Corollary 6.7. *Every proto-metrisable space is a D-space.*

Finally, let us mention a long standing problem of Borges and Wehrly. In [2], the authors asked whether monotonically normal paracompact spaces are D-spaces. Almost twenty years past, this question remains open. The following implications can be proved; for details see [15], [11] and [6].

metrisable ⇒ (linearly-)stratifiable ⇒ **elastic** ⇒
⇒ well-ordered (F) ⇒ monotone normal and (her.) paracompact

Since we know that elastic spaces are D-spaces, we think the following question is valuable to study.

Problem 6.8. *Are well-ordered (F) spaces D-spaces?*

We mention that Y. Z. Gao, H. Z. Qu and S. T.Wang gave an interesting characterization for monotonically normal paracompact spaces in [9].

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